

On the Stability of Friedrichs' Scheme and the Modified Lax-Wendroff Scheme*

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Abstract. Necessary and sufficient stability criteria for Friedrichs' scheme and the modified Lax-Wendroff scheme with smooth coefficients are derived by means of Kreiss' Matrix Theorem and the first Stability Theorem of Lax and Nirenberg.

In this note we derive necessary and sufficient stability criteria for Friedrichs' scheme and the *modified* Lax-Wendroff scheme [8] for the hyperbolic system

$$(1) \quad u_i = \sum_{j=1}^n a_j(x) u_{x_j}$$

of first-order differential equations with *variable* coefficient matrices. Friedrichs' scheme

$$v(x, t + k) = S_h v(x, t)$$

is given by the difference operator

$$S_h = \frac{1}{2n} \sum_i [T_i + T_i^{-1}] + \frac{1}{2} \lambda \sum_i a_i(x) [T_i - T_i^{-1}] \equiv C_h + \lambda A_h,$$

with T_i representing the translation operator, by the amount h in the x_i direction. The symbol of S_h is the matrix

$$(2) \quad s(x, \xi) = \frac{1}{n} \left(\sum_i \cos \xi_i \right) I + i \lambda \sum_i a_i(x) \sin \xi_i \equiv c(\xi) I + i \lambda a(x, \sin \xi),$$

where $\lambda = k/h$, k being the time step and h the space mesh size, $\xi = \{\xi_1, \dots, \xi_n\}$, $\sin \xi = \{\sin \xi_1, \dots, \sin \xi_n\}$, and I is the identity matrix.

With the above notation, the *modified* Lax-Wendroff scheme [8] is given by

$$v(x, t + k) = L_h v(x, t),$$

where the operator

$$L_h \equiv I + \lambda A_h [C_h + \frac{1}{2} \lambda A_h]$$

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has the symbol

$$(3) \quad v(x, \xi) = I + i\lambda a(x, \sin \xi)[c(\xi)I + \frac{1}{2}i\lambda a(x, \sin \xi)].$$

A scheme is stable if $\|v(T)\| \leq M(T)\|v(0)\|$, where M is a function of T but independent of h . The norm is $\|v(t)\|^2 = \int |v(x, t)|^2 dx$.

Kreiss' Matrix Theorem [1] states that a matrix $s(x, \xi)$ is stable if and only if there exists a positive-definite matrix $r(x, \xi)$ such that

$$s^*rs \leq r, \quad \text{for all } x, \xi.$$

If s is normal, $ss^* = s^*s$, Kreiss' Theorem reduces to von Neumann's condition with $r = I$.

We shall prove our stability theorem under either one of the following assumptions.

(A) The matrices $a_i(x)$ are *hermitian* and twice continuously differentiable.

(B) The matrices $a_i(x)$ are infinitely differentiable, and constant for $|x| > R$; moreover the matrix

$$a(x, \xi/|\xi|) = \sum_j a_j(x)\xi_j/|\xi|$$

admits a *positive-definite* symmetrizer $r(x, \xi)$, which is infinitely differentiable in x , and $\xi \neq 0$.

If the system (1) is *strictly hyperbolic* it follows by a result of Kreiss that such an r exists:

LEMMA [2]. *A matrix a admits a positive-definite symmetrizer r , $ra = a^*r$, if and only if the eigenvalues of a are real and its eigenvectors are linearly independent. In that case r can be written in the form $r = b^*b$ where $d = bab^{-1}$ is a diagonal matrix.*

In fact, strict hyperbolicity of (1) means that the eigenvalues of $a(x, \xi/|\xi|)$ are real and distinct for all x , and $\xi \neq 0$; hence the homogenized eigenvectors, $w_i(x, \xi/|\xi|)$ i.e., the rows of $b(x, \xi)$, are linearly independent and depend smoothly on x , and $\xi \neq 0$. It follows that the symmetrizer r is smooth in x , and $\xi \neq 0$.

Thus, under assumption (B) and by Kreiss' Lemma, the diagonal matrix

$$(4) \quad d(x, \xi) = b(x, \xi)a(x, \xi/|\xi|)b^{-1}(x, \xi)$$

is real. Moreover d and b are smooth in x and $\xi \neq 0$, independent of x for $|x| > R$, and homogeneous of degree zero in ξ ; hence they are symbols of pseudo-difference schemes of Yamaguti and Nogi [9]. Such schemes G_h with symbols $g(x, \xi)$ are defined by the formula

$$(G_h u)(x) = \int e^{ix \cdot \xi} g(x, \lambda_1(h\xi)) \hat{u}(\xi) d\xi,$$

where we take $\lambda_1(\xi) \equiv \sin \xi$. For these schemes we have the following form of the Lax-Nirenberg Theorem [7]: If the matrix $g(x, \xi)$ is a hermitian and nonnegative symbol of the pseudo-difference scheme G_h , then

$$\text{Re} (G_h \Lambda_h^2 u, u) \geq -Kh \|u\|^2.$$

Here Λ_h is the scheme with symbol $|\lambda_1(\xi)|$ and K is a constant independent of h .

THEOREM. *Under assumption (A) or (B), Friedrichs' scheme S_h and the modified Lax-Wendroff scheme L_h are stable if and only if their symbols s and l are stable.*

The following corollary gives sufficient conditions in terms of the absolute value of the largest eigenvalue of a , $|\sigma(a)|$.

COROLLARY. Under assumption (A) or (B), S_h is stable for those λ satisfying

$$(5) \quad \lambda |\sigma(a(x, \xi))| \leq n^{-1/2},$$

for all x and $|\xi| = 1$. Similarly L_h is stable for λ satisfying

$$(6) \quad \lambda |\sigma(a(x, \xi))| \leq 2n^{-1/2}.$$

Proof. We first prove the theorem and the corollary under assumption (A). In this case, the normal matrices s and l are stable if and only if

$$(7) \quad |\sigma(s(x, \xi))| \leq 1, \quad |\sigma(l(x, \xi))| \leq 1.$$

Since each of these inequalities expresses simultaneously von Neumann's necessary condition and the Lax-Nirenberg sufficient condition [3], [4] for the stability of S_h and L_h , the theorem follows. We deduce the corollary from the following inequalities

$$(8) \quad p \equiv I - s^*s \geq 0,$$

$$(9) \quad q \equiv I - l^*l \geq 0.$$

Using the identity

$$(10) \quad \begin{aligned} 1 - c^2(\xi) &= 1 - \left(\frac{1}{n} \sum_i \cos \xi_i\right)^2 \\ &= \frac{1}{n} \sum_i \sin^2 \xi_i + \frac{1}{n^2} \sum_{i>k} (\cos \xi_i - \cos \xi_k)^2, \end{aligned}$$

we obtain

$$\begin{aligned} p &\equiv p(\lambda) = I - c^2(\xi)I - \lambda^2 a^2(x, \sin \xi) \\ &= \frac{1}{n} |\sin \xi|^2 I + \frac{1}{n^2} \sum_{i>k} (\cos \xi_i - \cos \xi_k)^2 I - \lambda^2 a^2(x, \sin \xi) \\ &\geq \left\{ \frac{1}{n} I - \lambda^2 \left[\sum_i a_i(x) \frac{\sin \xi_i}{|\sin \xi|} \right]^2 \right\} |\sin \xi|^2 \\ &\geq \left\{ \frac{1}{n} - \lambda^2 \left| \sigma \left(a \left(x, \frac{\sin \xi}{|\sin \xi|} \right) \right) \right|^2 \right\} |\sin \xi|^2 I, \end{aligned}$$

whence (5) implies (8). Similarly, we find

$$q \equiv q(\lambda) = \lambda^2 a^2 [I - c^2 I - \frac{1}{4} \lambda^2 a^2] = \lambda^2 a^2 p(\lambda/2).$$

Thus (6) implies (9). Second, we consider assumption (B). By Kreiss' Lemma and Kreiss' Theorem the symbols s and l given by (2) and (3) respectively are stable if and only if

$$(11) \quad p_1 \equiv b_1^* b_1 - s^* b_1^* b_1 s \geq 0,$$

$$(12) \quad q_1 \equiv b_1^* b_1 - l^* b_1^* b_1 l \geq 0.$$

Here, $b_1 = b(x, \sin \xi)$. With $d_1 = d(x, \sin \xi)$, (10) and (4) give for (11) and (12)

$$(13) \quad \begin{aligned} p_1 &\equiv p_1(\lambda) \\ &= b_1^* \left\{ \frac{1}{n} I - \lambda^2 d_1^2 + \sum_{i>k} \left[|\sin \xi|^{-1} \left(\sin^2 \frac{\xi_i}{2} - \sin^2 \frac{\xi_k}{2} \right) \right]^2 I \right\} b_1 |\sin \xi|^2 \geq 0 \end{aligned}$$

and

$$(14) \quad a_1 \equiv a_1(\lambda) = \lambda^2 a_1^* p_1(\lambda/2) a_1 |\sin \xi|^2 \geq 0,$$

with $a_1 = a(x, \sin \xi / |\sin \xi|)$.

Since $a(x, \xi/|\xi|)$, b and d are symbols of pseudo-difference schemes and the terms in the summation $\sum_{i>k}$ in (13) admit the calculus of these schemes, we can apply the Lax-Nirenberg Theorem [7] to $p_1 \geq 0$ in (13) to obtain

$$(15) \quad \operatorname{Re} (P_{1,h} u, u) \geq -Kh \|u\|^2,$$

with

$$P_{1,h} u(x) = \int e^{ix \cdot \xi} p_1(x, h\xi) \hat{u}(\xi) d\xi.$$

It is shown in [9] that (15) implies the stability of S_h . Similarly (14) implies the stability of L_h . This proves the sufficiency of (11) and (12). The necessity of (11) and (12) is a consequence of Kreiss' Theorem and Kreiss' Lemma under assumption (B). The corollary follows also by the same theorem of Lax and Nirenberg [7].

This proves the theorem and the corollary.

If we require only Lipschitz continuity in assumption (A), the Lax-Wendroff Theorem [5] yields only strict inequality in (5) and (6) because it does not admit equality in (7). Whether the Lax-Nirenberg Theorem [3], [6] holds or not for Lipschitz coefficients is an open question. Assumption (B) on the $a_i(x)$ could be considerably weakened. The stability of L_h for strictly hyperbolic systems (1) with *constant* coefficients was treated by Wendroff [8].

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